On the Calculation of Nonlinear Spinor Field Functionals. III

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The limitations of the Green functions method concerning nonlinear spinor theory are discussed. To remove these difficulties, functional quantum theory of the nonlinear spinor field was introduced in preceding papers. To obtain numerical values for the global observables in this theory, the functional eigenstates have to be calculated. In this paper especially for functional scattering states appropriate equations are derived. A general integration method of the spinor field functional equation is introduced, leading to an equation for the irreducible part of the state functional. Generating functionals are defined, allowing a separation of selfenergy and interactionenergy terms in the equation of the irreducible part. By spectral decomposition of the scattering functionals the boundary conditions are examined, which lead in connection with selfenergy and interactionenergy terms to the construction of channel equations for the irreducible part of elastic scattering functionals. The method is extented to inelastic processes, it can be tested in the case of nonrelativistic scattering theory. The procedure for the three particle case is discussed in some details.

Nonlinear spinor theory treated by Heisenberg and coworkers [1, 2] is an approach to a unified microscopic description of matter. According to Heisenberg, for such a description it is sufficient to calculate the global observables, i.e. the quantum numbers of stationary states and the S-matrix [3]. Therefore, in nonlinear spinor theory at least these global observables have to be derived. But this task is not trivial. As the spinor field operators act in an indefinite state space and cannot be identified with special free matter fields of conventional quantum field theories, the methods of construction of observables provided by these theories do not suffice to solve this task correctly for the nonlinear spinor field itself. This has already been emphasized in preceding papers [4–7]. To remove this difficulty, Stumpf proposed a calculation program for the global observables [5] which led to the concept of functional quantum theory [8, 9]. The essential point of this approach is a map of the ordinary Hilbert space of quantum field theory into a corresponding functional Hilbert space, where the observables of interest can be defined. To obtain numerical values for these functional quantum observables comparable with experiment, the eigenstates of the nonlinear spinor field in the corresponding functional space have to be calculated. These spinor field functional states are especially bound functional states, describing single particles, and scattering functional states, describing the scattering of several particles. In preceding papers I [10] and II [11] a functional formalism for the construction of functional bound states and functional scattering states has been proposed. Although the direct explicit calculation of bound states is a very difficult task from a fundamental point of view, it is easy compared with the calculational problem of scattering states. The reason for this is the fact that the boundary conditions for scattering states are much more complicated than those of bound states. From the general formalism follows that the bound state problem is a subroutine of the scattering problem. Therefore, we concentrate on the discussion of scattering problems. At the end of II it was emphasized that a further elaboration of the boundary conditions and the channel construction prescription is necessary. This will be performed in this paper up to the point where practical calculations can be started.

Treating this problem, it should be noted that by Heisenberg, Dürr and coworkers for the calculation of global observables the Green functions method in combination with L.S.Z.-technic is preferred. As they are mainly interested in other topics of the spinor field problem, not very much work has been done so far by them in this direction. But the Green function approach is problematic as has already been stressed in some papers by Stumpf [4–7]. Its difficulty was one of the reasons for developing functional quantum theory. To justify our approach, we discuss in the first section the limitations of the Green functions method with respect to the spinor field problem before going on to treat our own problem. References to other comparable work will be given in the text. We shall demonstrate that there is a tight connection to the treatment of the multiparticle problem in nonrelativistic Schrödinger rep-

resentation, as it has been advanced by Faddeev [12], Sandhas [13], Wildermuth [14], Schmid [15] et al. The only structural difference between the nonrelativistic and the relativistic case is the occurence of indefinite metric in nonlinear spinor theory. Applying the unitarization method of Sudarshan [16], Karowski [17] et al. to functional quantum theory, there seems to be no serious obstacle in obtaining meaningful numerical values by relativistic calculation analogous to the nonrelativistic procedures. The main effort in functional quantum theory was directed to the goal to give a well founded basis for the application of this analogy. In its essential features this development is concluded by this paper.

1. Limitations of Green Functions Method

In conventional quantum field theories the S-matrix can be calculated by the L.S.Z.-formalism which is based on the use of the Green functions [18,19]. Concerning nonlinear spinor theory, Stumpf has repeatedly emphasized [4–7] that the application of the L.S.Z.-formalism in this case is problematic. To circumvent the difficulties connected with this formalism, Stumpf and coworkers developed the functional quantum theory as an alternative calculation scheme, which is applicable also to those cases where the L.S.Z.-formalism fails. In order to get a better understanding of the dynamics of the nonlinear spinor field, it is nevertheless desirable to analyse the difficulties of the L.S.Z.-formalism in nonlinear spinor theory in more detail. Generally, these difficulties are connected with indefinite metric and with the occurence of bound states. Concerning bound states, the situation was unsatisfactory for a long time. Only very simple field theoretic bound state models were treated which did not reach the level of complication occuring in nonlinear spinor theory. For a discussion see [6]. Considering the bound state description by Green functions method by analysing the multiparticle structure of Green functions, it was the aim to establish a Green function theory of multiparticle dynamics by Symanzik [20], Broido [20] et al. But there was no clear connection between L.S.Z.-formalism and Green functions formalism. Rather the T-matrices were given by definition. First attempts were made in nonrelativistic many particle theory to obtain a more rigorous deduction [22], but they still were not satisfactory. Due to the quark-model and the parton model in recent years the general interest in bound state problems increases. In a remarkable paper Huang and Weldon [23] were the first who gave a deduction of L.S.Z.-formalism on that level of complication which is characteristic of nonlinear spinor theory. This paper just discusses the problems which have been raised by Stumpf. Therefore, it is reasonable to begin with two formulae which were derived in a preceding paper [6], namely (1.34), (1.38). In nonlinear spinor theory any single real physical particle is described by being a bound state of the nonlinear spinor field. Its eigenstate $|k\rangle$ is represented by

$$|k\rangle = \sum_{n=\varrho}^{\infty} \sigma_n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n}|k)\, T\, \Psi_{\alpha_1}(x_1)\dots\Psi_{\alpha_n}(x_n)|0\rangle \tag{1.1}$$

where $\varrho \geqq 1$ is a definite number depending on the group theoretical properties of $|k\rangle$. Describing a system of interacting elementary (or not elementary) particles on the level of coupling field theories and on the level of nonlinear spinor theory, the S-matrix of the coupling field description can be transferred into the spinor field description giving there [6]

$$\begin{aligned} S(k_1\dots k_n|k_1'\dots k_m') :=& \\ \times \operatorname{Lim}^{-}_{n_1\dots n_n} \sum \sigma^x_{n_1}(z_1^1\dots z_{n_1}^1|t_1 k_1)\dots\sigma^x_{n_n}(z_1^n\dots z_{n_n}^n|t_n k_n)\times& \\ = \operatorname{Lim}^{+}_{n'_1\dots n'_m} \sum \sigma_{n_1'}(y_1^1\dots y_{n_1'}^1|t_1' k_1')\dots& \\ \times \sigma_{n'_m}(y_1^m\dots y_{n'_m}^m|t_m' k_m')\times& \\ \times\langle 0|(T\,\Psi(z_1^1)\dots\Psi(z_{n_1}^1))\dots& \\ \times(T\,\Psi(y_1^m)\dots\Psi(y_{n_m}^m))|0\rangle& \end{aligned} \tag{1.2}$$

with $\operatorname{Lim}^{+} := \lim_{\substack{t_1'\to\infty \\ \vdots \\ t_m'\to\infty}}$, $\operatorname{Lim}^{-} := \lim_{\substack{t_1\to-\infty \\ \vdots \\ t_n\to-\infty}}$.

Denoting the coefficients σ_n of the expansion (1.1) wave functions and especially the σ_n for $n \geqq \varrho+1$ the wave functions of the polarization clouds of the particles under consideration, Huang and Weldon put the following questions:

- α) Are there distinguished wave functions in (1.2)?
- β) What is the influence of the polarization cloud in (1.2)?
- γ) How can (1.2) be calculated?

The treatment of these questions contains the program of a rigorous foundation of multiparticle dynamics and a rigorous deduction of the connection between L.S.Z.-formalism and Green functions. This program is examplified by Huang and Weldon for locally coupled fermion-boson fields. The results

are: α) It is sufficient to use any suitably normalized wave function σ_ϱ provided, it has the correct transformation properties. β) Without any restriction to generality, the wave functions of the polarization cloud $\sigma_n, n \geqq \varrho+1$ can be omitted. γ) The special form of the wave function σ_ϱ drops out. Concerning γ) a more systematic treatment can be given: Performing a multiparticle analysis of the lowest Green function, dynamical equations for the connected parts etc. can be derived. Substitution of this expansion of G into the L.S.Z.-formula directly leads to the identification of the connected parts of the Green function with the T-matrix elements etc. This has been partly done in [22] for the three particle case.

The paper of Huang and Weldon shows that for a wellfounded application of L.S.Z.- and Green functions-technic in nonlinear spinor theory the same questions have to be answered. This fact has so far been ignored by Heisenberg, Dürr and coworkers. So only some uncritical papers on this subject have been given [24, 25]. Since it is the aim of this paper to develop a completely other approach to the S-matrix construction, it is not the intention to give here an extensive treatment on the pattern of Huang and Weldon. But we rather give an outline which demonstrates that the situation in nonlinear spinor theory is more difficult. Essentially two types of spinor theories have been used so far, namely Heisenberg's dipole regularized spinor theory and the polregularized theory of Dürr [26], Dammeier [27] et al. As both theories lead to different degrees of difficulties, we treat them separately.

Dipole-regularized Spinor Theory

Firstly, we discuss question α). We try to understand in a simple way, how it is possible to use any suitable wave function σ_ϱ and not a distinguished one. From (1.1) follows for the scalarproduct of two states $|b\rangle$, $|a\rangle$ the formula

$$\langle b \,|\, a\rangle = \sum_{n=1}^{\infty} \sigma_n(\overset{\alpha_1}{x_1} \ldots \overset{\alpha_n}{x_n} \,|\, a)\, \tau_n(\underset{\alpha_1}{x_1} \ldots \underset{\alpha_n}{x_n} \,|\, b) \times \tag{1.3}$$

with the usual definition of the τ_n-functions. The right side of (1.3) can be considered to be the representation of the scalar product in functional space. By the calculation methods used in nonlinear spinor theory the states (1.1) cannot be derived directly. Rather only the components $\{\tau_n\}$ are calculable. It is now essential in which way the states (1.1) can be constructed. In this case the group theory is very useful. If we assume that the set of states $\{|a\rangle\}$ are group theoretically completely nondegenerate, then the orthonormality relations between proper eigenstates of the spinor field follow already from their mere property of being irreducible base vectors of the corresponding representation, i.e. the internal dynamical structure of the wave function is irrelevant concerning orthonormality. For global observables constructed only by the scalar product of the eigenstates this means that the internal structure drops out. Concerning the states (1.1) from this follows that irrespectively of the internal structure coordinates of $\tau_j(x_1 \ldots x_j \,|\, a)$ a wave function $\sigma_j(x_1 \ldots x_j \,|\, a)$ can be chosen, where the only condition for σ_j is to satisfy the proper transformation properties, as already by this condition orthonormality is secured. Considering e.g. a two fermion bound state $\langle 0|T\psi(x_1)\varphi(x_2)|b\rangle$ the mass m of this state gives rise to an isolated pole in Huang and Weldon's approach and therefore, the argument for an arbitrary choice of σ_j works. Additionally, one can assume $\sigma_2 \neq 0$ while $\sigma_n \equiv 0$, $n \geqq 3$ as no new information turns out by making other assumptions. Therefore, for nondegeneracy one can understand, how Huang and Weldon's model works. But this mechanism breaks down for a degenerate spectrum. In nonlinear spinor theory a bound state $\langle 0|T\Psi(x_1)\Psi(x_2)|b\rangle$ of two fermions is in the region of fermion-ghost and ghost-ghost (or dipole ghost) scattering. In this case, the special form of the wave function cannot be chosen arbitrarily, but is of particular interest. As one knows, the orthonormality is achieved in the degenerate spectrum by different boundary conditions. But the incorporation of boundary conditions practically means a complete determination of σ_2. Additionally, we consider production processes. If fermions produce e.g. pions in this case a projection of a fermion state $\{\tau_j(F)\}$ on a fermion-boson state $\{\tau_j(F,B)\}$ is required. But this means that one is not allowed to use $\sigma_1(F) \neq 0$ for the fermion state only, as by this assumption no production at all would be possible. From this we conclude that all higher σ_j, $j \geqq \varrho+1$ contribute, and that for all higher σ_j the same arguments are valid concerning their determination with respect to boundary conditions. As the theory has to be uniquely defined, it is also not allowed to use for different purposes different representations of the σ_j. This means that

one has to construct explicitly (1.1). But in doing this, the entire intention of Green functions method is ruined, as in connection with the L.S.Z.-formalism the construction of state representations should be avoided. Therefore, it is valid to ask, whether Green functions method is of any use at all in the dipole spinor theory. E.g. one may try to apply multiparticle analysis by substituting the expressions in the L.S.Z.-formula and performing formally the integrations of the σ with the corresponding parts of the Green functions. This method does not work for production processes, as so far the decompositions of the Green functions are all symmetric. Additionally, the argument of Mitter has to be observed [28], as for $\varrho_0 = 0$ the conventional decomposition technic with its conventional interpretation cannot be used, cp. also Dammeier [27]. So there seems no good chance of application of this technic in this case.

Pole-regularized Spinor Theory

All arguments given for the dipole theory are still valid with one exception: as $\varrho_0 \neq 0$, the conventional decomposition technic can be applied.

Summarizing the discussion, we may state: the results of Huang and Weldon are not applicable to nonlinear spinor theory. Pure nucleonic processes can be performed by L.S.Z.-technic in the pole regularized theory, as has been demonstrated already by Dammeier [29]. For other scattering processes, one should try to substitute the Green functions decompositions into (1.2) and to perform the integrations with the σ formally, but this fails for production processes, as in this case the numerical relations between the different σ are of importance. Additionally, this way is possible only for the pole regularized theory.

Still, we have not yet mentioned the unitarization procedure. Adopting the standpoint of Sudarshan and Karowski, this procedure can be performed in the state space directly, i.e. in functional quantum theory, but gives rise to new confusions in Green functions method. First of all, it is supposed that this unitarization procedure works so far only for dipole ghosts. But then the $\varrho_0 = 0$ argument prevents the use of Green functions. Additionally, as from the arguments of Karowski follows, in the theory only the S_F-propagators have to occur and nothing else. But this again is an unclear point.

By these arguments one recognizes that the application of Green functions and L.S.Z.-technic is a doubtful procedure and one should look for alternative methods, a part of which is treated in this paper.

2. Elastic Scattering Equations

To avoid lengthy repetitions, we frequently refer to the preceding papers I and II, as well as to other essential papers on functional quantum theory. The functional eigenstates of the nonlinear spinor field are given by the time ordered functional states, which can be written generally [II]

$$|\mathfrak{T}(j,a)\rangle := \sum_{n=1}^{\infty} \tau_n(x_1 \ldots x_n \,|\, a)\,|D_n(x_1 \ldots x_n)\rangle \quad (2.1)$$

where the set $\{|D_n\rangle\}$ defines the base functionals of the functional space and $|a\rangle$ means the corresponding eigenstate in ordinary Hilbert space. The quantum numbers characterizing the functional eigenstates due to the relativistic invariance of the theory are given by the equations [II]

$$\begin{aligned} \mathfrak{P}_h\,|\mathfrak{T}(j,a)\rangle &= p_h\,|\mathfrak{T}(j,a)\rangle\,, \\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\mathfrak{T}(j,a)\rangle &= s(s+1)\,|\mathfrak{T}(j,a)\rangle\,, \\ \mathfrak{P}^2\,|\mathfrak{T}(j,a)\rangle &= m^2\,|\mathfrak{T}(j,a)\rangle\,, \\ \mathfrak{S}_3\,|\mathfrak{T}(j,a)\rangle &= s_3\,|\mathfrak{T}(j,a)\rangle\,. \end{aligned} \quad (2.2)$$

For simplicity we do not refer to additional quantum numbers resulting from additional symmetry groups. (2.1) has to satisfy the dynamical functional equation [7]

$$[\gamma^\mu\,\partial_\mu\,\partial(x) + V\,\partial(x)\,\partial(x)\,\partial(x)]\,|\mathfrak{T}(j,a)\rangle = 0 \quad (2.3)$$

resulting from the nonlinear spinor field equation in ordinary Hilbert space. To define the regularizing noncanonical quantization, the normal transformed functional

$$|\Phi(j,a)\rangle := \exp[j(x)F(x,y)j(y)]\,|\mathfrak{T}(j,a)\rangle \quad (2.4)$$

has to be used in the following. It can be written

$$|\Phi(j,a)\rangle = \sum_{n=1}^{\infty} \varphi_n(x_1 \ldots x_n \,|\, a)\,|D_n(x_1 \ldots x_n)\rangle\,. \quad (2.5)$$

Under the normal transformation Eq. (2.2) remain invariant, giving [II]

$$\begin{aligned} \mathfrak{P}_h\,|\Phi(j,a)\rangle &= p_h\,|\Phi(j,a)\rangle\,, \\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\Phi(j,a)\rangle &= s(s+1)\,|\Phi(j,a)\rangle\,, \\ \mathfrak{P}^2\,|\Phi(j,a)\rangle &= m^2\,|\Phi(j,a)\rangle\,, \\ \mathfrak{S}_3\,|\Phi(j,a)\rangle &= s_3\,|\Phi(j,a)\rangle\,, \end{aligned} \quad (2.6)$$

while the dynamical Eq. (2.3) goes over into [7]

$$[\gamma^\mu \partial_\mu \mathrm{d}(x) + V\,\mathrm{d}(x)\,\mathrm{d}(x)\,\mathrm{d}(x)]\,|\,\Phi(j,a)\,\rangle = 0 \quad (2.7)$$

with $\mathrm{d}(x) := \partial(x) + F(x,x')\,j(x')$. Defining the projection operators

$$P_k := k!\,|\,D_k(x_1 \ldots x_k)\,\rangle\,\langle D_k(x_1 \ldots x_k)\,|\;\; 1 \leqq k < \infty \quad (2.8)$$

for a given eigenstate $|\,\Phi(j,a)\,\rangle$ there exists a smallest $k \equiv \varrho$ for which $P_k\,|\,\Phi(j,a)\,\rangle \neq 0$, $\varrho \leqq k < \infty$ while $P_k\,|\,\Phi(j,a)\,\rangle = 0$, $1 \leqq k \leqq \varrho - 1$ is valid. In I and II a general solution procedure was developed, where the Eq. (2.7) is reduced to an equation for $P_\varrho\,|\,\Phi(j,a)\,\rangle$ only. This step is common to bound state calculations as well as to scattering calculations. While in bound state calculations the equation for $P_\varrho\,|\,\Phi(j,a)\,\rangle$ is the final equation, in scattering calculations an additional step has to be performed, namely to derive from the equation for $P_\varrho\,|\,\Phi(j,a)\,\rangle$ appropriate channel equations, which is not a trivial task. In I and II for the reduction procedure projection operators have been used. As will be seen in the next section, this method is not very suitable for the subsequent derivation of the channel equations in the scattering case. Therefore, it is necessary to reformulate the reduction procedure by a method without using projection operators. This is the aim of this section.

We define

$$|\,\Phi_k(j,a)\,\rangle := P_k\,|\,\Phi(j,a)\,\rangle\,, \quad 1 \leqq k < \infty\,. \quad (2.9)$$

Then (1.5) can be written

$$|\,\Phi(j,a)\,\rangle = \sum_{l=0}^{\infty} |\,\Phi_{\varrho+l}(j,a)\,\rangle \quad (2.10)$$

where ϱ can be fixed uniquely in dependence of the quantum numbers of the eigenstate $|a\rangle$, like angular momentum, baryon number etc. Substitution of (2.10) into (2.7) and applying $P_{\varrho+l}$ to (2.7) gives the set of equations

$$\begin{aligned}&\delta_x\,\partial_x\,|\,\Phi_{\varrho+l}\rangle + \delta_x F_x j\,|\,\Phi_{\varrho+l-2}\rangle + V\,\partial_x{}^3\,|\,\Phi_{\varrho+l+2}\rangle\\ &\quad + 3\,V F_x j\,\partial_x{}^2\,|\,\Phi_{\varrho+l}\rangle + 3\,V (F_x j)^2\,\partial_x\,|\,\Phi_{\varrho+l-2}\rangle\\ &\quad + V (F_x j)^3\,|\,\Phi_{\varrho+l-4}\rangle = 0\\ &\qquad l = 2\nu\,, \quad 0 \leqq \nu < \infty\,, \;\; \nu \text{ integer}\end{aligned} \quad (2.11)$$

with the abbreviations

$$\delta_x := \gamma^\mu \partial_\mu\,; \quad \partial_x := \partial(x)\,; \quad (2.12)$$
$$F_x j := F(x,x')\,j(x')\,; \quad d_x := d(x)\,; \quad j_x := j(x)\,.$$

(2.11) is valid for arbitrary ϱ and reads in a compact notion

$$\sum_{l=0,2,4\ldots} O_{k,l}(x,\partial,j)\,|\,\Phi_{\varrho+l}(j,a)\,\rangle = 0\,, \quad k = 0,2,4\ldots \quad (2.13)$$

as the operators occuring in (2.11) are independent of ϱ. We introduce the new notation

$$|\,\varphi_\varrho\rangle := |\,\Phi_\varrho\rangle\,; \quad |\,\chi_{\varrho+l}\rangle := |\,\Phi_{\varrho+l+2}\rangle\,. \quad (2.14)$$

Then with the same operator O_{kl} (2.11) can be formulated equivalently by

$$\begin{aligned}&\delta_x\,\partial_x\,|\,\varphi_\varrho\rangle + V\,\partial_x{}^3\,|\,\chi_\varrho\rangle + 3\,V F_x j\,\partial_x{}^2\,|\,\varphi_\varrho\rangle = 0\,,\\ &\sum_{l=0,2,4\ldots} O_{k,l}(x,\partial,j)\,|\,\chi_{\varrho+l}\rangle = -\,\delta_{0k}\,3\,V (F_x j)^2\,\partial_x\,|\,\varphi_\varrho\rangle\\ &\quad - \delta_{0k}\,\delta_x F_x j\,|\,\varphi_\varrho\rangle - \delta_{2k}\,V (F_x j)^3\,|\,\varphi_\varrho\rangle, \quad k = 2\nu\,.\end{aligned} \quad (2.15)$$

Defining the auxiliary functional

$$|\,\chi(j,a)\,\rangle := \sum_{l=0,2,\ldots} |\,\chi_{\varrho+l}\rangle\,. \quad (2.16)$$

Equations (2.15) can be rewritten into the compact form

$$\begin{aligned}&(\delta_x\,\partial_x + 3\,V F_x j\,\partial_x{}^2)\,|\,\varphi_\varrho\rangle + V\,\partial_x{}^3\,|\,\chi\rangle = 0\,,\\ &(\delta_x\,d_x + V\,d_x\,d_x\,d_x)\,|\,\chi\rangle\\ &\quad = -\,[3\,V (F_x j)^2\,\partial_x + \delta_x F_x j + V (F_x j)^3]\,|\,\varphi_\varrho\rangle\end{aligned} \quad (2.17)$$

under the auxiliary condition to apply P_ϱ to the first equation. Therefore, to perform the first step in the reduction procedure namely to eliminate the polarization cloud given by $|\,\chi\rangle$, the Green functional of the original Eq. (2.7) has to be constructed. To achieve this, we look for an iterative solution of the equation

$$(\delta_x\,d_x + V\,d_x\,d_x\,d_x)\,|\,\chi\rangle = |\,j_x\rangle\,. \quad (2.18)$$

Defining

$$\delta_x\,d_x\,|\,\chi\rangle =: |\,K_x\rangle \quad (2.19)$$

we may write (2.18) with $G_{xx'}\,\delta_{x'} = \mathbf{1}$

$$|\,K_x\rangle + V\,d_x\,d_x\,G_{xx'}\,|\,K_{x'}\rangle = |\,j_x\rangle\,. \quad (2.20)$$

The iterative solution of (2.20) is given by

$$|\,K_x\rangle = \sum_\nu (V\,d_x\,d_x\,G_{xx'})^\nu\,|\,j_{x'}\rangle\,. \quad (2.21)$$

To obtain the final solution we observe

$$j_x\,|\,K_x\rangle = j_x\,\delta_x\,d_x\,|\,\chi\rangle = i\,\gamma^\mu\,\mathfrak{P}_\mu\,|\,\chi\rangle\,. \quad (2.22)$$

Due to the fact that the momentum operator $\mathfrak{P}_\mu$ defines good quantum numbers, this operator commutes with any operator of Eq. (2.3) as well as of Equation (2.7). Writing the first equation of (2.17)

in the equivalent form

$$(\delta_x d_x + V d_x d_x d_x)\,|\varphi_\varrho\rangle + V\,\partial_x{}^3\,|\chi\rangle = 0 \quad (2.23)$$

we obtain by multiplying (2.23) with $i\gamma^\mu \mathfrak{P}_\mu$

$$(\delta_x d_x + V d_x d_x d_x)\, i\gamma^\mu \mathfrak{P}_\mu\,|\varphi_\varrho\rangle + V\,\partial_x{}^3 j_x\,|K_x\rangle = 0 \quad (2.24)$$

and by substitution of (2.21)

$$\begin{aligned}&(\delta_x d_x + V d_x d_x d_x)\, i\gamma^\mu \mathfrak{P}_\mu\,|\varphi_\varrho\rangle \\ &\quad - V\,\partial_x{}^3 j_x\,[\textstyle\sum_\nu (V d_x d_x G_{xx'})^\nu] \qquad (2.25)\\ &\quad [3V(F_{x'}j)^2\,\partial_{x'} + \delta_{x'}F_{x'}j + V(F_{x'}j)^3]\,|\varphi_\varrho\rangle = 0\\ &\quad =: O[j,\partial]\,|\varphi_\varrho\rangle = 0\end{aligned}$$

which is the reduced equation we intended to derive.

The essential feature of this technic is that the kernels of the relevant equations are derived by an iteration procedure, while the solutions, of course, have to be obtained by noniterative technics. This method is in complete analogy with the Green functions treatment, where the kernels of the Green function equations are constructed by iterative technic, too, cp. [22, 28].

3. Generating Channel Functionals

As has been already emphasized by general theoretical reasons as well as by the reasons of practical calculations, an explicit representation of the channel equations for scattering functionals is required. This problem will be discussed in this section. As has been shown in II, in functional quantum theory like in ordinary quantum mechanics the introduction of channels produces an antisymmetry breaking. Therefore, the construction of the full antisymmetric scattering states has to be performed by using virtual channel scattering states with a subsequent superposition restoring full antisymmetry. In this intermediate step of channel definition and calculation we are allowed therefore to break antisymmetry. It is just this fact which leads to an explicit calculation of channel equations. To do this, we consider Eq. (2.25) and the corresponding functional state $|\varphi_\varrho\rangle$ with

$$|\varphi_\varrho\rangle := \varphi_\varrho(x_1 \dots x_\varrho\,|\,a)\,|D_n(x_1 \dots x_\varrho)\rangle\,. \quad (3.1)$$

To define a channel state, we introduce a partition of $x_1 \dots x_\varrho$, namely

$$\begin{aligned}&x_1 \dots x_\varrho \to \qquad (3.2)\\ &(x_1 \dots x_{\varrho_1}\,|\,x_{\varrho_1+1} \dots x_{\varrho_1+\varrho_2}\,|\dots|\,x_{\varrho_1+\dots\varrho_{i-1}+1} \dots x_{\varrho_i})\end{aligned}$$

which corresponds to a division of $\varphi_{\varrho'}(x_1 \dots x_\varrho)$ into n cluster states with ϱ_i, $1 \leqq i \leqq n$ coordinates. Within any cluster, the coordinates are assumed to be antisymmetric in the corresponding state functions, while between different cluster coordinates no antisymmetry is present. As has been shown in II, the full antisymmetry can then be restored by a superposition of n clusters with dimensions ϱ_i $1 \leqq i \leqq n$, where the partitions are taken of suitable permutations of $x_1 \dots x_\varrho$. Formally we express this by an operator $Q_\varkappa(n)$. Having derived a state function

$$\begin{aligned}&\varphi^n_\varrho := \qquad (3.3)\\ &\varphi_\varrho(x_1 \dots x_{\varrho_1}\,|\,x_{\varrho_1+1} \dots x_{\varrho_1+\varrho_2}\,|\dots|\,x_{\varrho_1+\dots\varrho_{n-1}+1} \dots x_{\varrho_n})\end{aligned}$$

then the totally antisymmetric state function is given by

$$\begin{aligned}&\varphi_\varrho(x_1 \dots x_\varrho) \qquad (3.4)\\ &\quad = \sum_\varkappa Q_\varkappa(n)\,\varphi_\varrho(x_1 \dots x_{\varrho_1}\,|\dots|\,x_{\varrho_1+\dots\varrho_{n-1}+1} \dots x_\varrho)\,.\end{aligned}$$

Having derived the channel equation for one representative n-cluster channel, the equations of permuted channels follow by application of $Q_\varkappa$. Thus we can restrict ourselves to the discussion of one representative channel for a n-cluster system with the partition $\varrho_1 \dots \varrho_n$. For simplicity we chose as a representative partition for this case (3.2). As for (3.2) the state functions (3.3) are antisymmetric only for the internal groupings of the n cluster, it is reasonable to use corresponding auxiliary functional spaces which reflect this property. This can be achieved by introducing n different source operators j_ϱ, $1 \leqq \alpha \leqq n$ and ∂_α, $1 \leqq \alpha \leqq n$ acting in different functional spaces $\mathfrak{H}_\alpha$ with

$$\begin{aligned}&[j_\alpha(x), j_\alpha(x')]_+ = [\partial_\alpha(x)\,\partial_\alpha(x')]_+ = 0\\ &[j_\alpha(x), \partial_\alpha(x')]_+ = \delta(x-x') \qquad (3.5)\end{aligned}$$

while the operators from different spaces $\mathfrak{H}_\alpha$ commute. Forming then the direct product of these spaces $\mathfrak{H} = \mathfrak{H}_1 \otimes \dots \otimes \mathfrak{H}_n$ the base functionals are given by

$$\begin{aligned}&|D_n(x^1_1 \dots x^1_{z_1}\,|\dots|\,x^n_1 \dots x^n_{z_n})\rangle \qquad (3.6)\\ &\quad = |D_{\alpha_1}(x^1_1 \dots x^1_{z_1})\rangle \otimes \dots \otimes |D_{z_n}(x^n_1 \dots x^n_{z_n})\rangle\end{aligned}$$

and corresponding orthonormality relations can be derived. By means of these base functionals, a chan-

nel functional of a n-cluster state can be written

$$|\varphi_\varrho^n\rangle := \varphi_\varrho(x_1^1 \dots x_{\varrho_1}^1 | \dots | x_1^n \dots x_{\varrho_n}^n) \times |D_\varrho(x_1^1 \dots x_{\varrho_1}^1 | \dots | x_1^n \dots x_{\varrho_n}^n)\rangle . \tag{3.7}$$

Now the channel equations can be derived in the following way: We substitute in (2.25) for ∂_x, d_x and j_x the expressions

$$\partial_x \to \sum_{\alpha=1}^{n} \partial_{\alpha x}, \quad d_x \to \sum_{\alpha=1}^{n} d_{\alpha x}, \quad j_x \to \sum_{\alpha=1}^{n} j_{\alpha x}. \tag{3.8}$$

Then the functional operator of (2.25) is given by

$$\begin{aligned} &[\delta_x(\Sigma d_{\alpha x}) + V(\Sigma d_{\alpha x})^3](\Sigma j_{\alpha x})\delta_x(\Sigma \partial_{\alpha x}) \\ &\quad - V(\Sigma \partial_{\alpha x})^3(\Sigma j_{\alpha x})\,[\Sigma_\nu (V(\Sigma d_{\alpha x})^2 G_{xx'})^\nu] \\ &\quad \times [3V(F_{x'}(\Sigma j_{\alpha x})^2(\Sigma \partial_{\alpha x'}) + \delta_{x'} F_{x'}(\Sigma j_\alpha) \\ &\qquad + V(F_{x'}(\Sigma j_\alpha))^3 \\ &\quad \equiv O[\Sigma j_\alpha, \Sigma \partial_\alpha] = \Sigma_\alpha O[j_\alpha, \partial_\alpha] + \text{mixed terms}. \end{aligned} \tag{3.9}$$

As for a single eigenstate, the operator of (2.25) is defined by

$$K_{\varrho_i}(x_1^i \dots x_{\varrho_i}^i | y_1^i \dots y_{\varrho_i}^i) := \langle D_{\varrho_i}(x_1^i \dots x_{\varrho_i}^i) | O[j_i, \partial_i] | D_{\varrho_i}(y_1^i \dots y_{\varrho_i}^i)\rangle . \tag{3.10}$$

We obtain the channel equation

$$\begin{aligned} &\langle D_\varrho(x_1^1 \dots x_{\varrho_1}^1 | \dots | x_1^n \dots x_{\varrho_n}^n) | O[\Sigma j_\alpha, \Sigma \partial_\alpha] | D_\varrho(y_1^1 \dots y_{\varrho_1}^1 | \dots | y_1^n \dots y_{\varrho_n}^n)\rangle \varphi_\varrho(y_1^1 \dots y_{\varrho_1}^1 | \dots | y_1^n \dots y_{\varrho_n}^n) \\ &= [\sum_{i=1}^{n} K_{\varrho_i}(x_1^i \dots x_{\varrho_i}^i | y_1^i \dots y_{\varrho_i}^i) + V(x_1^1 \dots x_{\varrho_1}^1 | \dots | x_1^n \dots x_{\varrho_n}^n | y_1^1 \dots y_{\varrho_1}^1 | \dots | y_1^n \dots y_{\varrho_n}^n)] \\ &\qquad \times \varphi_\varrho(y_1^1 \dots y_{\varrho_1}^1 | \dots | y_1^n \dots y_{\varrho_n}^n) = 0 \end{aligned} \tag{3.11}$$

where V is the matrix element of the mixed terms in (3.9). From this equation the permuted equations follow according to II

$$[\sum_i Q_x^n K_{\varrho_i} Q_x^{n-1} + Q_x^n V Q_x^{n-1}]\, Q_x^n \varphi_\varrho(y_1^1 \dots y_{\varrho_1}^1 | \dots | y_1^n \dots y_{\varrho_n}^n) = 0 \tag{3.12}$$

and the final solution is given by (3.4). Having derived an explicit expression for the channel equations, we have to investigate, in which way the boundary conditions for the scattering functionals can be incorporated into the scattering equation. In the next section we discuss by spectral decomposition the general structure of the scattering state function, while in the fifth section the resulting boundary conditions are explicitly incorporated. It can be shown that in ordinary nonrelativistic quantum theory the derivation of channel equations by generating channel functionals just reproduces the well known nonrelativistic channel equations.

4. Spectral Cluster Decomposition

To explore the asymptotic behaviour of scattering states, a spectral decomposition of the state functions resp. state functionals has to be made. In preceding papers such a spectral decomposition has been given [30]. It reads for a n-point function

$$\begin{aligned} \tilde{\tau}_n(q_1 \dots q_n | \mu_n) = &\sum_{\mu_1 \dots \mu_{n-1}} \sum_{\lambda_1 \dots \lambda_n} (-1)^P M(\mu_1 \dots \mu_n)(2\pi)^{3n+1} i^{n-1} \\ &\times \delta(\sum_{s=1}^{n} q_s - p_{\mu_n}) \prod_{s=1}^{n-1} \delta(\sum_{r=1}^{s} q_{\lambda_r} - p_{\mu_s})(\sum_{r=1}^{s} q_{\lambda_r}^0 - p_{\lambda_r}^0 + i\varepsilon)^{-1} \end{aligned} \tag{4.1}$$

with

$$M(\mu_1 \dots \mu_n) := \langle 0 | \Psi(0) | \mu_1\rangle \dots \langle \mu_{n-1} | \Psi(0) | \mu_n\rangle . \tag{4.2}$$

As this decomposition follows only from Poincaré invariance and timeordering it is true generally. But for an analysis of asymptotic scattering states it is still not sufficiently adapted. This can be easily seen by analyzing the intermediate states p_{μ_i}, $1 \leqq i \leqq n-1$. Disregarding other quantum numbers, these intermediate states can be classified by their mass and their momentum. Denoting the masses occuring in the intermediate states $|\mu_j\rangle$ by m_α^j, $1 \leqq \alpha \leqq r_j$ it may happen that the mass spectra for different intermediate state families $\{|\mu_j\rangle\}$, $1 \leqq j \leqq n-1$ are partially or completely different. This indicates that compared with the elementary fermions occuring in $\{|\mu_1\rangle\}$ other particles i.e. new bound states and, of course, scattering states appear. Thus to obtain a meaningful spectral decomposition of the state functionals, these different mass spectra have to be taken into account additionally. This will be done in this section.

To perform this, we write for brevity in a selfexplanatory way

$$\tau_\varrho(x_1 \dots x_\varrho \,|\, k_a) \equiv \sum_{\lambda_1 \dots \lambda_n} \langle 0 \,|\, \lambda_1 \dots \lambda_\varrho \,|\, k_a \rangle \, (-1)^P \, \Theta(\lambda_1 - \lambda_2) \dots \Theta(\lambda_{\varrho-1} - \lambda_\varrho) \tag{4.3}$$

and we denote a partition of $(x_1 \dots x_\varrho)$ into n clusters by

$$\text{Part } (x_1 \dots x_\varrho) \rightarrow (i_1^1 \dots i_{\varrho_1}^1) \dots (i_1^n \dots i_{\varrho_n}^n) \tag{4.4}$$

with $\sum_{j=1}^{n} \varrho_j = \varrho$ and with $x_{i_\beta^\alpha} \rightarrow i_\beta^\alpha$. Then it is possible to express the total permutation of ϱ coordinates by

$$\sum_{\lambda_1 \dots \lambda_\varrho} (\lambda_1 \dots \lambda_\varrho) = \sum_\varkappa Q_\varkappa \Big(\sum_{\lambda_1^1 \dots \lambda_{\varrho_1}^1} (i_{\lambda_1^1}^1 \dots i_{\lambda_{\varrho_1}^1}^1) \dots \sum_{\lambda_1^n \dots \lambda_{\varrho_n}^n} (i_{\lambda_1^n}^n \dots i_{\lambda_{\varrho_n}^n}^n) \Big) \tag{4.5}$$

where $Q_\varkappa$ means the permutation operator of Section 2.

Defining now the expression

$$\begin{aligned} R_\varkappa(k_1 \dots k_n) :&= \sum_{\lambda_1^1 \dots \lambda_{\varrho_1}^1} \langle 0 \,|\, i_{\lambda_1^1}^1 \dots i_{\lambda_{\varrho_1}^1}^1 \,|\, k_1 \rangle \, \Theta(\lambda_1^1 - \lambda_2^1) \dots \Theta(\lambda_{\varrho_1 - 1}^1 - \lambda_{\varrho_1}^1) \, (-1)^{p_1} \dots \\ &\quad \times \sum_{\lambda_1^n \dots \lambda_{\varrho_n}^n} \langle 0 \,|\, i_{\lambda_1^n}^n \dots i_{\lambda_{\varrho_n}^n}^n \,|\, k_n \rangle \, \Theta(\lambda_1^n - \lambda_2^n) \dots \Theta(\lambda_{\varrho_{n-1}}^n - \lambda_{\varrho_n}^n) \, (-1)^{p_n} \\ &\quad \times \Theta(\lambda_{\varrho_1}^1 - \lambda_1^2) \, \Theta(\lambda_{\varrho_2}^2 - \lambda_1^3) \dots \Theta(\lambda_{\varrho_{n-1}}^{n-1} - \lambda_1^n) \end{aligned} \tag{4.6}$$

we may write

$$\sum_{\lambda_1 \dots \lambda_\varrho} \langle 0 \,|\, \lambda_1 \dots \lambda_\varrho \,|\, k_a \rangle \, \Theta(\lambda_1 - \lambda_2) \dots \Theta(\lambda_{\varrho-1} - \lambda_\varrho) \, (-1)^p = \sum_{\text{Part } n = 1 \dots \varrho} \sum_\varkappa Q_\varkappa(n) \sum_{k_1 \dots k_{n-1}} R_\varkappa(k_1 \dots k_n)/k_n \equiv k_a \tag{4.7}$$

where the different partitions just sort out these masses resp. poles, which do not occur in the forgoing one. Thus starting with $R(k_1 \dots k_\varrho)$, this partition is that part of the expansion (4.1) where $\{(k_i - k_{i-1})^2\} = \{m_i^2\}$ coincides with the mass spectrum of $\{k_1^2\} = \{m_1^2\}$, while the other partitions contain masses resp. poles, which are just not contained in $\{m_1^2\}$. The sum in (4.1) therefore can be decomposed into the partition sums

$$\sum_{\mu_1 \dots \mu_{\varrho-1}} = \sum_{\substack{\mu_1 \dots \mu_{\varrho-1} \\ (p_i - p_{i-1})^2 = \{m_i^2\} \equiv \{m_1^2\} \\ i = 1 \dots \varrho}} + \sum_{\substack{\mu_1 \dots \mu_{n-1} \\ \text{Part } (p_i - p_{i-1})^2 = \{m_i^2\} \equiv \{m_1^2\} \\ i = 1 \dots \varrho - 1}} + \dots \,. \tag{4.8}$$

Thus expansion (4.7) is equivalent expansion (4.1) by introducing in (4.1) the various partition classes. In the following we consider first the various partition terms (3.6). The Fourier-transform of (4.6) reads

$$\begin{aligned} &\tilde{R}_\varkappa(k_1 \dots k_n) \\ &= \int \sum_{\lambda_1^1 \dots \lambda_{\varrho_1}^1} \langle 0 \,|\, i_{\lambda_1^1}^1 \dots i_{\lambda_{\varrho_1}^1}^1 \,|\, k_1 \rangle \, \Theta(\lambda_1^1 - \lambda_2^1) \dots \Theta(\lambda_{\varrho_1 - 1}^1 - \lambda_{\varrho_1}^1) \, (-1)^{p_1} \exp\Big\{ -i \sum_{j=1}^{\varrho_1} p_j^1 \, i_{\lambda_j^1}^1 \Big\} \, d\, i_{\lambda_1^1}^1 \dots d\, i_{\lambda_{\varrho_1}^1}^1 \dots \\ &\quad \times \int \sum_{\lambda_1^n \dots \lambda_{\varrho_n}^n} \langle k_{n-1} \,|\, i_{\lambda_1^n}^n \dots i_{\lambda_{\varrho_n}^n}^n \,|\, k_n \rangle \, \Theta(\lambda_1^n - \lambda_2^n) \dots \Theta(\lambda_{\varrho_n - 1}^n - \lambda_{\varrho_n}^n) \, (-1)^{P_n} \exp\Big\{ -i \sum p_j^n \, i_{\lambda_j^n}^n \Big\} \, d\, i_{\lambda_1^n}^n \dots d\, i_{\lambda_{\varrho_n}^n}^n \\ &\quad \times \int (u_1 + i\varepsilon)^{-1} \dots (u_{n-1} + i\varepsilon)^{-1} \exp\{ i\, u_1 (i_{\lambda_{\varrho_1}^1}^1 - i_{\lambda_1^2}^2) \dots i\, u_{n-1} (i_{\lambda_{\varrho_{n-1}}^{n-1}}^{n-1} - i_{\lambda_1^n}^n) \} \, d\, u_1 \dots d\, u_{n-1} \,. \end{aligned} \tag{4.9}$$

By

$$i_{\lambda_k^h}^h = z^h + z_{\lambda_k^h}^h, \quad 1 \leqq k \leqq \varrho_h, \qquad \text{with} \quad \sum_{j=1}^{\varrho_h} z_{\lambda_j^h}^h = 0 \tag{4.10}$$

and

$$d\, i_{\lambda_1^h}^h \dots d\, i_{\lambda_{\varrho_h}^h}^h = d\, z_{\lambda_1^h}^h \dots d\, z_{\lambda_{\varrho_h - 1}^h}^h d\, Z^h \tag{4.11}$$

we introduce center of mass coordinates in (4.9). By a suitable denumeration of the variables and by integration over Z^h, $1 \leqq h \leqq n$ we obtain

$$\begin{aligned} \tilde{R}_\varkappa(k_1 \dots k_n) = \int \sum_{\lambda_1^1 \dots \lambda_{\varrho_1}^1} &\langle 0 \,|\, z_1^1 \dots z_{\varrho_1 - 1}^1 \,|\, k_1 \rangle \, \Theta(z_1^1 - z_2^1) \dots \Theta(z_{\varrho_1 - 1}^1 - z_{\varrho_1}^1) \, (-1)^{P_1} \, \delta\Big(\sum_{j=1}^{\varrho_1} z_j^1 \Big) \\ &\times \exp\Big\{ -i \sum_{j=1}^{\varrho_1} p_{\lambda_j^1}^1 \, z_j^1 \Big\} \, d\, z_1^1 \dots d\, z_{\varrho_1}^1 \, \delta(\mathfrak{R}_1 - \mathfrak{P}_1) \, \delta(k_1^0 - P_1^0 + u_1) \end{aligned} \tag{4.12}$$

$$\times \int \sum_{\lambda_1^n \dots \lambda_{\varrho_n}^n} \langle k_{n-1} | z_1^n \dots z_{\varrho_n - 1}^n | k_n \rangle \, \Theta(z_1^n - z_2^n) \dots \Theta(z_{\varrho_n - 1}^n - z_{\varrho_n}^n) \, (-1)^{P_n} \, \delta\Big(\sum_{j=1}^{\varrho_n} z_j^n\Big)$$

$$\times \exp\Big\{-i \sum_{j=1}^{\varrho_n} p_{\lambda_j^n}^n \, z_j^n\Big\} \, d z_1^n \dots d z_{\varrho_n}^n \, \delta(\Re_n - \Re_{n-1} - \mathfrak{P}_n) \, \delta(k_n^0 - k_{n-1}^0 - P_n^0 + u_{n-1})$$

$$\times \int (u_1 + i\varepsilon)^{-1} \dots (u_{n-1} + i\varepsilon)^{-1} \exp\{i u_1 (z_{\varrho_1}^1 - z_1^2)\} \dots \exp\{i u_{n-1} (z_{\varrho_n - 1}^{n-1} - z_1^n)\} \, d u_1 \dots d u_{n-1} .$$

Performing an expansion of this expression at the poles we obtain

$$\tilde{R}_\varkappa(k_1 \dots k_n) = \tilde{\tau}_{\varrho_1}(p_1^1 \dots p_{\varrho_1}^1 | k_1) \, \delta(k_1 - P_1) \dots \tilde{\tau}_{\varrho_n}(p_1^n \dots p_{\varrho_n}^n | k_n - k_{n-1}) \, \delta(k_n - k_{n-1} - P_n) + \tilde{r}_\varkappa(k_1 \dots k_n) \tag{4.13}$$

where $r_\varkappa(k_1 \dots k_n)$ is a regular rest term and we use the definition for the Fourier transform

$$\tilde{\tau}_{\varrho_i}(p_1^i \dots p_{\varrho_i}^i | k_i - k_{i-1}) := \tilde{\mathfrak{S}} \langle k_{i-1} | T \, x_1^i \dots x_{\varrho_i}^i | k_i \rangle . \tag{4.14}$$

If the particles are far away, we may put

$$\langle k_{i-1} | T \, x_1^i \dots x_{\varrho_i}^i | k_i \rangle \approx \langle 0 | T \, x_1^i \dots x_{\varrho_i}^i | k_i - k_{i-1} \rangle \tag{4.15}$$

i.e. the "groundstate" $|k_{i-1}\rangle$ acts on the particle $(k_i - k_{i-1})$ like the true vacuum state $|0\rangle$. Thus we have shown that the partition $\varkappa(n)$ contains in its pole term just the state functionals of the cluster partition $\varkappa(n)$. Observing that a scattering state $|k_n^{(\pm)}\rangle$ can be written

$$|k_n^{(\pm)}\rangle = |k_n, \mathrm{sc}\rangle + |k_1\rangle \otimes |k_1 - k_2\rangle \otimes \dots \otimes |k_n - k_{n-1}\rangle \tag{4.16}$$

we may assume that the matrix element of the ingoing resp. outgoing state is just given by

$$\tilde{\tau}_\varrho(p_1 \dots p_\varrho | k_1 \otimes k_2 - k_1 \otimes \dots \otimes k_n - k_{n-1}) = \tilde{R}_\varkappa(k_1 \dots k_n) + r_\varkappa + \dots \tag{4.17}$$

while all other partitions disappear by projection of the field operators on the asymptotic product states. Therefore, by Eq. (4.17) the asymptotic conditions for the channel $Q_\varkappa(n)$ are explored, which was the intention of this discussion. It should be mentioned that this result coincides with the assumptions made about the asymptotic behaviour of state functionals in the preceding paper II.

5. Inelastic Scattering Equations

As has been already noticed in the first section, the method of deriving scattering equations given there is restricted to elastic scattering processes. We demonstrate this for a simple example. Considering nucleon-nucleon scattering, the lowest nonvanishing amplitude is $\varrho = 2$. Therefore, the equation of the nucleon-nucleon scattering amplitude has to be calculated on $M_4 \otimes M_4$. Raising the energy up to the level of pion production, we obtain simultaneously $n+n \to n+n$ but also $n+n \to n+n+\pi$. For $n+n+\pi$ the lowest nonvanishing amplitude is $\varrho = 4$. Therefore, by opening the channel of one pion production, we need at least an amplitude defined on $M_4 \otimes M_4 \otimes M_4 \otimes M_4$. On the other hand, the ingoing configuration may be only $n+n$. The resolution of this difficulty is to consider $n+n$ for $\varrho = 4$. In this case the generating channel functional for incoming nucleons has to be considered on the sector (3/1) resp. (1/3). Therefore, one nucleon has to be described by a threepoint function, while the other still has to be described by a onepoint function. In order to separate the interaction terms from the selfenergy terms one has therefore to look not only for an eigenvalue equation of a single nucleon for the onepoint function, but equivalently also an eigenvalue equation for the threepoint function. It will be shown in this section that it is possible to derive a functional eigenvalue equation which is valid for all orders, i.e. which gives applied to a certain n-point function always the corresponding equation in configuration space. We demonstrate this by deriving an eigenvalue equation for $|\varrho_{\varrho+2}\rangle$ instead of $|\varphi_\varrho\rangle$, which correspond just to the situation of describing a nucleon not by $\varphi_1(x)$ but by $\varphi_3(x_1, x_2, x_3)$. Using the notation of the first section, the general functional equation is given by (2.11). We introduce now the new notation

$$|\varphi\rangle := |\Phi_\varrho\rangle ; \quad |\xi\rangle := |\Phi_{\varrho+2}\rangle ; \quad |\chi_l\rangle := |\Phi_{\varrho+4+l}\rangle . \tag{5.1}$$

By introducing additionally

$$|\chi(j, a)\rangle := \sum_{l=0,2,4\dots} |\chi_l\rangle \tag{5.2}$$

the system of Eqs. (2.11) can then be written in the following way

$$\begin{aligned}
&\delta_x \partial_x |\varphi\rangle + V \partial_x^3 |\xi\rangle + 3 V F_x j \partial_x^2 |\varphi\rangle = 0 ,\\
&\delta_x \partial_x |\xi\rangle + V \partial_x^3 |\chi\rangle + 3 V F_x j \partial_x^2 |\xi\rangle + 3 V (F_x j)^2 \partial_x |\varphi\rangle + \delta_x F_x j |\varphi\rangle = 0 ,\\
&(\delta_x d_x + V d_x d_x d_x) |\chi\rangle = -[V (F_x j)^3 + 3 V (F_x j)^2 \partial_x + \delta_x F_x j] |\chi\rangle - V (F_x j)^3 |\varphi\rangle =: |g_x\rangle .
\end{aligned} \tag{5.3}$$

The general solution according to (2.21) reads

$$\delta_x \partial_x |\chi\rangle = \sum_\nu (V d_x d_x G_{xx'})^\nu |g_{x'}\rangle \tag{5.4}$$

or

$$i \gamma^\mu \mathfrak{P}_\mu |\chi\rangle = \sum_\nu j_x (V d_x d_x G_{xx'})^\nu |g_{x'}\rangle \tag{5.5}$$

Multiplying the $|\xi\rangle$-equation by $i \gamma^\mu \mathfrak{P}_\mu$ we obtain

$$\begin{aligned}
&[\delta_x \partial_x + 3 V F_x j \partial_x^2] i \gamma^\mu \mathfrak{P}_\mu |\xi\rangle\\
&\quad - V \partial_x^3 [\sum_\nu j_x (V d_x d_x G_{xx'})^\nu]\\
&\quad \times [V (F_{x'} j)^3 + 3 V (F_{x'} j)^2 \partial_{x'} + \delta_{x'} F_{x'} j] |\xi\rangle\\
&= \{V \partial_x^3 [\sum_\nu j_x (V d_x d_x G_{xx'})^\nu]\\
&\quad \times (V (F_x j)^3 - 3 V (F_x j)^2 \partial_x - \delta_x F_x j)\} |\varphi\rangle .
\end{aligned} \tag{5.6}$$

Finally, we eliminate $|\varphi\rangle$ by the same technic as applied in the first section. We obtain

$$i \gamma^\mu \mathfrak{P}_\mu |\varphi\rangle = -\sum_\mu j_x (3 V F_x j \partial_x G_{xx'})^\mu V \partial_{x'}^3 |\xi\rangle \tag{5.7}$$

and therefore by multiplying (5.6) with $i \gamma^\mu \mathfrak{P}_\mu$

$$\begin{aligned}
&\{[\delta_x \partial_x + 3 V F_x j \partial_x^2] \mathfrak{P}^2\\
&\quad - V \partial_x^3 [\sum_\nu j_x (V d_x d_x G_{xx'})^\nu]\\
&\times [V (F_{x'} j)^3 + 3 V (F_{x'} j)^2 \partial_{x'} + \delta_{x'} F_{x'} j] \gamma^\mu \mathfrak{P}_\mu\} |\xi\rangle\\
&= -\{V \partial_x^3 [\sum_\nu j_x (V d_x d_x G_{xx'})^\nu]\\
&\quad \times [V (F_{x'} j)^3 + 3 V (F_{x'} j)^2 \partial_{x'} - \delta_{x'} F_{x'} j]\\
&\quad \times \sum_\mu j_x (3 V F_x j \partial_x G_{xx'})^\mu V \delta_{x'}^3\} |\xi\rangle .
\end{aligned} \tag{5.8}$$

This is the final functional equation for processes with production amplitudes $|\Phi_{\varrho+2}\rangle$ mixed with $|\Phi_\varrho\rangle$. Considering this equation as a bound state equation, one can apply it to the lowest sector, but then follows that $V \partial_{x'}^3 |\xi\rangle$ vanishes. So (5.8) is a bound state equation, which is valid for $|\Phi_{\varrho_1}\rangle$ as well as for $|\Phi_{\varrho_1+2}\rangle$. Thus we have the possibility to define an ingoing or outgoing channel in terms of $\varphi_{\varrho_i+2}(x_1 \ldots x_{\varrho_i+2})$ as well as in terms of $\varphi_{\varrho_i}(x_1 \ldots x_{\varrho_i})$. Therefore, by observing that for $\varphi_4(x_1 \ldots x_4)$ above the threshold of pion production or nucleon-antinucleon production the following channels are open $n+n$, $n+n+\pi$, $n+n+n+\bar{n}$, $\pi+\pi$ we may treat the following initial or final configurations:

$$\begin{aligned}
n+n &=: \varphi_3{}^{\mathrm{W}}(x_1 x_2 x_3)\, \varphi_1{}^{\mathrm{W}}(x_4) ,\\
n+n+\pi &=: \varphi_1{}^{\mathrm{W}}(x_1)\, \varphi_1{}^{\mathrm{W}}(x_2) \varphi_2{}^{\pi}(x_3 x_4) ,\\
\pi+\pi &=: \varphi_2{}^{\pi}(x_1 x_2)\, \varphi_2{}^{\pi}(x_3 x_4) ,\\
n+n+n+\bar{n} &=: \varphi_1{}^{\mathrm{W}}(x_1)\, \varphi_1{}^{\mathrm{W}}(x_2)\, \varphi_1{}^{\mathrm{W}}(x_3)\, \varphi_1{}^{\mathrm{W}}(x_4) .
\end{aligned} \tag{5.9}$$

For these channels, Eq. (5.8) can be treated by the methods discussed in the previous sections. But one observes that the amount of formulating the channel equations increases enormously. Therefore, for high energy production processes it would be desirable to develop a variational treatment, where the channel configurations can be handled easier. Nevertheless, if one likes to apply the variational or projection methods, which have been successfully developed by Wildermuth and coworkers [14] for the nonrelativistic problem, one can start immediatly with the equations just derived using the scalar product formation given in a preceding paper [31]. But also the other nonrelativistic solution procedures can be transferred immediatly, as will be demonstrated in the following section.

6. Three Particle Scattering Formalism

By the foregoing considerations we have reached a far-reaching analogy to nonrelativistic multichannel scattering Schrödinger equations. To obtain from these multichannel Schrödinger equations numerical results, several successful formalisms have been developed, especially the socalled Faddeev procedure. Due to the analogy just mentioned it is reasonable to apply these procedures to the relativistic case. This will be done in this section for the extremely simple case of the threeparticle scattering, which in the relativistic notion only means that we consider an equation for $\varphi_3(x_1 x_2 x_3)$. This equation contains the $n+n+n$, $n+\pi$, $n+\bar{n}+n$, $\bar{n}+\bar{n}+n$, $\bar{n}+\bar{n}+\bar{n}$, $\bar{n}+\pi$ channels, of course also all channels, which arise by replacing π by a vector meson etc. For higher channels, the equations derived in Sect. 2 or 5 can also be used, but then the generalized Faddeev-method has to be used. As no production channels are assumed to be open, we use the elastic

Equation (2.25). We write it in the form

$$[\delta_x d_x \gamma^\mu \mathfrak{P}_\mu + \vartheta(x, j, \partial)] \,|\, \varphi_3\rangle = 0 \qquad (6.1)$$

with

$$\vartheta(x, j, \partial) := V d_x d_x d_x \gamma^\mu \mathfrak{P}_\mu - V \partial_x{}^3 j_x [\sum_\nu (V d_x d_x G_{xx'})^\nu] \qquad (6.2)$$
$$\times [3 V (F_{x'} j)^2 \partial_{x'} + \delta_{x'} F_{x'} j + V (F_{x'} j)^3] .$$

To obtain a complete analogy to the stationary nonrelativistic Schrödinger equation, we multiply (6.1) by j_x and observe (2.6). Then we obtain

$$-j_x \vartheta(x, j, \partial) \,|\, \varphi_3\rangle = m^2 \,|\, \varphi_3\rangle . \qquad (6.3)$$

Considering for this equation scattering solutions, the subsystems of nucleons resp. pions satisfy the bound state equations

$$-j_x \vartheta(x, j, \partial) \,|\, \varphi_1\rangle = m_{\mathrm{w}} \,|\, \varphi_1\rangle ; \qquad (6.4)$$
$$-j_x \vartheta(x, j, \partial) \,|\, \varphi_2\rangle = m_\pi^2 \,|\, \varphi_2\rangle .$$

It might be that these equations are not very well suited for bound state calculations, but in any way they are satisfied by the bound states, as they are necessary conditions for these states. On the other hand, having derived the bound states, Eq. (6.3) seems to be very well suited for scattering calculations, as the interaction term is not modified essentially by the procedure performed above. Applying the procedure of Sect. 2, one has to distinguish four channels, given by the equations

$$\langle D_3(x_\alpha \,|\, x_\beta x_\gamma) \,|\, -j_x \vartheta \,|\, D_3(y_\alpha \,|\, y_\beta y_\gamma) \rangle \varphi_3(y_\alpha \,|\, y_\beta y_\gamma) = m^2 \varphi_3(x_\alpha \,|\, x_\beta x_\gamma) , \quad a = 1, 2, 3 \qquad (6.5)$$

and

$$\langle D_3(x_1 \,|\, x_2 \,|\, x_3) \,|\, -j_x \vartheta \,|\, D_3(y_1 \,|\, y_2 \,|\, y_3) \rangle \varphi_3(y_1 \,|\, y_2 \,|\, y_3) = m^2 \varphi_3(x_1 \,|\, x_2 \,|\, x_3) \qquad (6.6)$$

which are special cases of (3.11) for the three particle channels. Applying the definitions (3.10) and (3.11) to equations (6.5), (6.6) we may write

$$[K_1(x_\alpha \,|\, y_a) + K_2(x_\beta x_\gamma \,|\, y_\beta y_\gamma) + V(x_\alpha \,|\, x_\beta x_\gamma \,|\, y_\alpha \,|\, y_\beta y_\gamma)] \varphi_3 = m^2 \varphi_3 \qquad (6.7)$$

and

$$[\sum_{a=1}^{3} K_1(x_a \,|\, y_a) + V(x_1 \,|\, x_2 \,|\, x_3 \,|\, y_1 \,|\, y_2 \,|\, y_3)] \varphi_3 = m^2 \varphi_3 . \qquad (6.8)$$

Then the channel resolvents can be written

$$g_a(z) := (z - K_1(x_a \,|\, y_a) + K_2(x_\beta x_\gamma \,|\, y_\beta y_\gamma))^{-1} \qquad (6.9)$$

and

$$g_0(z) := (z - \sum_{a=1}^{3} K(x_a \,|\, y_a))^{-1} \qquad (6.10)$$

with the corresponding interaction potentials

$$V_a := V(x_a \,|\, x_\beta x_\gamma \,|\, y_a \,|\, y_\beta y_\gamma) \qquad (6.11)$$

and

$$V_0 := V(x_1 \,|\, x_2 \,|\, x_3 \,|\, y_1 \,|\, y_2 \,|\, y_3) . \qquad (6.12)$$

For the free ingoing or outgoing configurations we obtain from (6.4)

$$K_1(x \,|\, y) \varphi_1(y) = m_{\mathrm{w}}^2 \varphi_1(x) ; \qquad (6.13)$$
$$K_2(x, x' \,|\, y, y') \varphi_2(y, y') = m_\pi^2 \varphi_2(x, x') .$$

Therefore, from (6.7) and (6.8) follows by inversion of the free channel operators

$$\varphi_\alpha^{(\pm)}(x_a \,|\, x_\beta x_\gamma \,|\, n) = \varphi_1(x_a \,|\, \nu_1) \varphi_2(x_\beta x_\gamma \,|\, \nu_2) + g_a(m^2 \pm i\varepsilon) V_a \varphi_\alpha^{(\pm)}(x_a \,|\, x_\beta x_\gamma \,|\, n) \qquad (6.14)$$

and

$$\varphi_0^{(\pm)}(x_1 \,|\, x_2 \,|\, x_3 \,|\, n) = \varphi_1(x_1 \,|\, \nu_1) \varphi_1(x_2 \,|\, \nu_2) \varphi_3(x_3 \,|\, \nu_3) + g_0(m^2 \pm i\varepsilon) V_0 \varphi_0^{(\pm)}(x_1 \,|\, x_2 \,|\, x_3 \,|\, n) \qquad (6.15)$$

with $n := \nu_1 \nu_2$ resp. $n = \nu_1 \nu_2 \nu_3$ the quantum numbers of the ingoing resp. outgoing configurations.

As has been shown by detailed investigations in the nonrelativistic case, the channel Lipmann Schwinger equations (6.14) resp. (6.15) are not suitable for practical calculations. They are neither unique nor integrable. To remove these difficulties, the Faddeev procedure has to be performed. As we have derived sofar the relativistic three particle equations in complete analogy to the nonrelativistic case, we assume that also the Faddeev procedure is meaningful for the relativistic equations. As the Faddeev procedure is a standard procedure, we do not derive it here. We give only its results. They are cp. [15]

$$\varphi_\alpha^{(\pm)}(n)_i = \delta_{i\alpha} \varphi_\alpha^{\mathrm{f}}(n) + \sum_{i=1}^{3} g_0(m^2 \pm i\varepsilon) \times F_{ij}(m^2 \pm i\varepsilon) \varphi_\alpha^{(\pm)}(n)_j$$
$$\varphi_\alpha^{(\pm)}(n) = \sum_{i=1}^{3} \varphi_\alpha^{(\pm)}(n)_i \qquad (6.16)$$

and

$$\varphi_0^{(\pm)}(n)_i = \varphi_i^{\mathrm{f}}(n) + \sum_{j=1}^{3} g_0(m^2 \pm i\varepsilon) \times F_{ij}(m^2 \pm i\varepsilon) \varphi_0^{(\pm)}(n)_j$$
$$\varphi_0^{(\pm)}(n) = \sum_{i=1}^{3} \varphi_0^{(\pm)}(n)_i \qquad (6.17)$$

with

$$F_{ij}(z) := (1 - \delta_{ij}) t_i(z) \qquad (6.18)$$

and $$t_i(z) := (1 - V_i g_0)^{-1} V_i . \qquad (6.19)$$

For further details see [15].

Concluding this section, some remarks have to be made concerning specialities of the nonlinear spinor theory. First of all, the eigenvalue equations (6.13) for the nucleon do not have only the nucleon eigensolutions, but also the good and bad ghost solutions, which afterwards are combined to the dipole pair. According to Stumpf and Scheerer [32] the bad ghost has to be identified with the tachyon. Performing the unitarization, not only outgoing nucleons have to be considered, but also outgoing tachyons, which raises the number of channels, Especially in $n-n$-scattering already different channels occur due to this situation. Concerning the situation of bound states with negative norm it is not clear, whether such states can be prevented at all or not. Therefore, the investigation on this subject still has to be accomplished. Secondly, from Bethe-Salpeter formalism non-Schrödinger like many particle equations are accustomed. So one may ask whether Equations (6.7), (6.8) give a sound basis for calculations with respect to the pole structure of the resulting solutions. Only speculations are possible. As the equations follow by rigorous deduction, they have to be satisfied in any case. Probably the proper pole structure is achieved just by applying Faddeev's procedure which guarantees unique solutions. But if unique solutions are obtained, they have to be the same as those resulting from the calculation procedure with other formalism. Finally, it should be mentioned that in the quark-model also relativistic invariant dynamical equations with Schrödinger structure, namely addition of the kinetic parts, have been introduced by Feynman and coworkers [33, 34].

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